Multipulse embedded solitons as bound states of quasi-solitons

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Abstract

'Embedded solitons' is a name given to localised solutions of multi-component dispersive wave models, that occur in resonance with the linear spectrum. They exist at isolated (codimension-one) parameter values within the steady state equations which have linearisation with eigenvalues $\pm\lambda$ and $\pm i\omega$. At the same time, quasi-solitons, or generalised solitary waves with non-decaying radiation tails, are known to be endemic. We consider the general question of when two quasi-solitons may be glued together to form a two-humped 'bound state' embedded soliton in the limit $\lambda/\omega \to 0$. A generalization of the method of Gorshkov and Ostrovsky is used within the framework of general normal forms for Hamiltonian reversible systems. A simple asymptotic formula is derived that governs the existence, symmetry and accumulation rate of the bound states. This generalises earlier ad hoc calculations for specific examples. The formula depends on few simple ingredients: the eigenvalues of the equilibrium, an asymptotic estimate for the tail amplitude of the quasi-solitons and the 'Birkhoff signature' and symmetry properties of the Hamiltonian. Only the tail amplitude estimate is difficult to calculate, requiring exponential asymptotics. But it is shown that this only affects the third-order term in the asymptotic formula. The calculation is worked out in detail for the steady states of several example PDE systems, taken from nonlinear optics and fluid mechanics; and excellent agreement with numerical results is found.

Key words: nonlinear optics, Hamiltonian system, embedded soliton, bound state, exponential asymptotic, homoclinic orbit, saddle centre equilibrium
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1 Introduction

In many multi-component or higher-order nonlinear dispersive wave models taking the form of partial differential equations (PDEs) in one space and one time dimension, there is the possibility of *embedded solitons* (ESs) [48]. These are so called because they are embedded into (i.e. in resonance with) the linear spectrum [1]. See for example the review [17]. Examples are known in models arising from nonlinear optics, such as second-harmonic generation (SHG) [48,8], three wave interaction [15] and a model for Bragg-grating solitons [16,14]. Also, examples have been found in nonlinear Schrödinger equations with higher-order perturbation terms [7,21,34] and in higher-order and coupled Korteweg de Vries equations that arise in the theory of water waves [24,11]. Embedded solitons are known to have several unusual properties. First, they are isolated solutions. That is, they are of codimension-one in the region of physical parameter space where there is resonance with linear waves. This is the so-called *saddle-centre* region, see Section 2 below, where the associated steady-state ordinary differential equation (ODE) system has linearisation consisting of both real eigenvalues $\pm\lambda$ and imaginary eigenvalues $\pm i\omega$. Second, an ES can be at most only weakly stable. That is they may be linearly neutrally stable, but nonlinearly have a one-sided algebraic instability [48,43] which in certain cases may be exponentially weak [49].

More generally, in the saddle-centre region one finds steady, almost localized, structures, *quasi-solitons* (QSs) or *homoclinics to periodic orbits*, which take the forms of a soliton core with a non-decaying radiating tail travelling at the same wave speed as the core, see e.g. [6]. A particular property of these solutions is that for an arbitrary fixed system parameter there exists a one-dimensional family of QSs which may be parametrised by the phase shift between the core and the oscillatory tail. Thus we might refer to such objects as being of 'codimension-minus-one' in the parameter space. Throughout this paper we shall be concerned with the case where the travelling-wave ODE is both Hamiltonian and reversible (see Section 2 below), and we shall be interested in the limit of a singular saddle-centre, that is when $\lambda/\omega \to 0$. Within this class of systems, quite general estimates exist for the one-dimensional family of QSs which may be written as a curve in the space of amplitude of the periodic tails of the QS versus the phase shift between the two tails [37]. In particular the minimum tail amplitude is an exponentially small function of λ/ω.

In general to estimate the factor C, say, multiplying this exponentially small

[1] Here, and throughout this work, the word soliton is used in an informal, physical context and strictly speaking we should refer to solitary waves, since the underlying mathematical models we study are non-integrable

estimate for the minimum tail amplitude, is a challenging task and involves asymptotic methods that go beyond all algebraic orders. In many examples which come from physical context (see e.g. [46,31,23,9] and [50]) the appearance of the imaginary eigenvalues $\pm i\omega$ is associated with a weak singular perturbation to a completely integrable Hamiltonian PDE. Then one has the existence formally of a small amplitude, long, sech or sech^2-like pulse coupled to the $O(1)$ radiation with frequency ω. Alternatively, upon rescaling, this can be considered to be an $O(1)$ pulse that excites radiation at high frequencies, $\omega \to \infty$, where the radiation is an exponentially small function in this limit. In the latter context, several methods have been used to estimate C; for example matched asymptotics in the complex plane together with Borel summation [44,46,2,23,25,24], or a perturbation procedure in the wave-number domain [31,9,50]. These approaches have been rigorously justified in the recent work of Lombardi [37] who uses the notation of perturbed normal forms of the associated ODEs, that we shall adopt in Section 2 below, and makes careful estimates near singularities of the solution in the complex plane. He proves that C is *generically non-zero* as a function of a single parameter.

Now, if this factor were to vanish, as a function of another parameter then this would correspond to the existence of a 'fundamental' or 'one-hump' ES. See for example the results of Grimshaw and Cook [24] who find curves of such zeros of C in the parameter space of a coupled KdV system by Borel summation. Also, in [10] a formal observation suggests that, if one considers the singular limit to be $\lambda \to 0$ while nonlinearity and ω remain $O(1)$, then for fixed λ, $C(\omega)$ may be periodic in some cases. This conclusion is backed up in [10] by numerics on various fourth-order ODE models.

However, even if C never vanishes, there is another possibility for the existence of ESs. That is, one might be able to glue together two QSs so that the radiation is precisely in anti-phase, in such a way that the glued object also solves the equation. Thus these two-humped ESs maybe thought of as 'bound-states' (BSs) of two QSs. If this is possible, then one could in principle also form multi-humped bound states with any number N of humps, but in this paper we shall exclusively consider the case $N = 2$. Henceforth we shall use the abbreviation BS for such objects. Note that there is growing numerical evidence that such solutions exist in a number of physical examples. For example, Buryak [7] found, for both a higher-order NLS equation and a SHG model, a sequence of BSs at parameter values that accumulate on the asymptotic limit $\lambda/\omega \to 0$. Other works [21,9,16,34] have found a similar accumulation in a variety of different physical models, including evidence (and in [9] a detailed asymptotic argument) that the rate of convergence to the singular limit is algebraic. These and other examples are revisited in Section 4 below.

The aim of this paper is to provide a general answer to the question of whether such BSs exist for a particular model, and if so what are their symmetry properties and their asymptotics in the singular limit $\lambda/\omega \to 0$. We make use extensively of the asymptotic method due to Gorshkov and Ostrovsky [22], adapting it to the case of non-zero tail radiation. This approach is justified only in the limit of well-separated one-pulse QS solutions, where the overlap between neighboring QS cores is small. Then, the slow evolution of the BS can be described in terms of classical mechanics where the interaction potential is defined through the non-quadratic part of the Hamiltonian of the PDEs. Stationary BSs realize extrema of this interaction potential when a delicate balance between attraction and repulsion is obtained. This condition shows that a steady solution of the PDEs exist, which implies a solution of the corresponding ODE system. This requirement is combined with the condition that the tails are in anti-phase so that the tails of the two QSs exactly cancel each other. Section 3 provides the details of this asymptotic calculation.

We note that we only present a "limited" generalisation of the Gorshkov and Ostrovsky approach from solitons to quasi-solitons, adapting it to find stationary quasi-soliton BSs (i.e. two-hump localised solutions of certain ODE systems). Calculation of dynamical behaviour of far-separated QSs due to interaction forces (which is an essential topic of the original Gorshkov and Ostrovsky theory) is a PDE-related issue and is beyond our current analysis. This includes the stability analysis of QS BSs which we also leave for further investigation.

In order to present a general calculation, we use the notation of normal forms of reversible Hamiltonian systems of two-degrees of freedom. Using centre-manifold theory, our results are likely to apply in cases where the traveling wave system is higher-dimensional also, where the additional spectrum is off the imaginary axis (see e.g. [37,50] for an estimate of the factor C above in the infinite-dimensional steady water-wave problem), but we do not pursue such issues here. Our main result is to provide a universal formula that gives the asymptotics of a sequence of parameter-values converging *algebraically* on the singular limit. Its leading-order terms depend only on certain simple properties of the linearised system and the symmetry of the entire system. In particular, we show that the factor C that is so hard to calculate, and is the only part of the formula that differs according to the precise nonlinearity of the model in question, enters the asymptotic expression as the third-order term. In Section 4, we find an excellent agreement with numerical results for a number of different models including the prediction of cases where BSs *do not* exist. Before proceeding to present our results, we should mention two previous results in the literature which have inspired this study.

First, we mention the work of Mielke, Holmes and O'Reilly [42]. They provide a rigorous proof of the existence of BS of ESs (not QSs as in this paper) under a condition on sign of the so-called Birkhoff signature of the quadratic part of the Hamiltonian. That is, if this condition is met, then for each one-pulse ES existing at a parameter value $\mu = 0$ say [*away* from the singular limit $(\lambda/\omega) \to 0$], there will be sequences of parameter values at which there exist two-pulse ESs. These sequences converge *exponentially* to the branch of fundamental ESs. Their result is more general and also shows the existence of N-pulses for all $N > 2$ and indeed chaotic dynamics of the initial-value problem. This Birkhoff signature condition appears explicitly in our result too. It explains the negative result for fifth-order KdV and in systems with odd symmetry it predicts whether the BSs we find by our method are of even or odd type.

Second, we mention the asymptotic analysis of Calvo and Akylas [9] (and the earlier results of [35]) which, like this paper, is concerned with BSs in the singular limit $\lambda/\omega \to 0$. Rather than present a general formula, they present a method that is then worked out in detail for two examples; a NLS equation perturbed with a third order dispersive term, and the 5th-order KdV equation. For the perturbed NLS example, they go further than us in that they construct asymptotic expressions for three and four-humped BSs. For the 5thKdv they find no bound states of the kind considered here (i.e. with zero tail radiation). In contrast, although we restrict attention to two-humped BSs, our results are more general in that we provide a *universal asymptotic formula*; equations (3.23), (3.43) or (3.54) depending only the symmetry properties of the system. This formula is valid for *all* examples and trivial to calculate up to its second-order term. It also explains in a nutshell the negative result for the 5th-order KdV equation and other examples for which the Birkhoff signature is of the wrong sign.

2 Reversible two degree of freedom Hamiltonian systems

We consider systems that are derived from nonlinear wave equations which can be written in Hamiltonian form as

$$u_\tau = \mathcal{J}\nabla\mathcal{H}(u) \tag{2.1}$$

for a wave profile $u(z,\tau)$ which is most generally a complex-valued vector function of a spatial co-ordinate $z \in \mathbb{R}$ and time $\tau \in \mathbb{R}^+$. Here $\mathcal{J}$ is a skew-symmetric operator and $\mathcal{H}$ is the PDE Hamiltonian. To study solitons we consider, by moving into a moving frame if necessary, solutions that are steady states of (2.1). That is, $u = x(z)$, where x solves an ODE system

$$\dot{x} = f(x,\mu) \tag{2.2}$$

where a dot represents differentiation with respect to z which, for convenience we henceforth refer to as ‘time’ t, and μ is a vector of parameters. In this paper we shall be specifically concerned with the case where the ODE (2.2) is fourth-order, Hamiltonian (in a finite-dimensional sense) and reversible. [2] That is, without loss of generality after a change of co-ordinates, we assume that

$$x \in \mathbb{R}^4, \quad f(x) = J\nabla H(x), \quad Rf(x) = -f(Rx) \tag{2.3}$$

with

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and the reversing transformation R acting on the 'momentum variables';

$$R : (x_1, x_2, x_3, x_4) \to (x_1, -x_2, x_3, -x_4). \tag{2.4}$$

The ODE (2.2), (2.3) inherits the structure of nonlinear terms from that of the underlying PDE (2.1). For the wide class of models where the generalised kinetic energy is a quadratic form, the above mentioned property corresponds to a simple relation between generalised PDE and ODE potentials. In particular, in the asymptotic analysis in the next section we shall explicitly use the relation

$$\mathcal{H}^{non-quadratic} = k \int_{-\infty}^{+\infty} H^{non-quadratic} dt, \tag{2.5}$$

valid for real stationary solutions of PDE system. In relation (2.5) $k = -1$ if the PDE (2.1) is defined on the class of real-valued vector functions u and $k = -2$ for the complex-valued u.

Now, we shall be interested in the case where the origin of Eq. (2.2) is an equilibrium of saddle-centre type, that is

$$f(0) = 0 \quad \sigma(Df(0)) = \{\pm\lambda, \pm i\omega\}.$$

Note that, again without loss of generality by a change of co-ordinates if necessary, this is equivalent to the assumption that the leading-order part of the Hamiltonian H can be written in one of two equivalent ways. We explicitly choose the so-called type-I (or $0^{2+}i\omega$) case in the notation of Lombardi [37]. That is, the quadratic part of the Hamiltonian is written as

$$H^{lin} = \frac{1}{2}(x_2^2 - \lambda^2 x_1^2) - \frac{\omega}{2}(x_3^2 + x_4^2) + O(|x|^3). \tag{2.6}$$

As argued in [10], such a choice describes the physical phenomenon we are interested in; where a 'sech-like' pulse is perturbed by 'fast' radiation. This

[2] Such systems arise in studies of higher-order Hamiltonian PDEs or a system of two coupled second-order PDEs, see Section 4 for several examples.

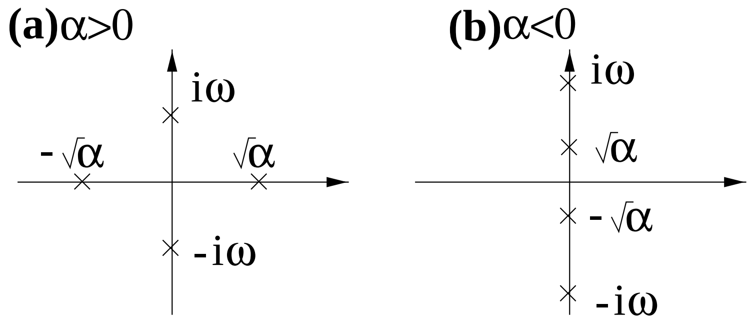

Fig. 1. Assumed behaviour of the linearisation about the origin under perturbation by $\alpha = \lambda^2$.

leads to a linearisation of (2.2) of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & \lambda^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{2.7}$$

Observe that changing the sign of ω in (2.6) does not change the eigenvalues of the equilibrium, it merely changes the location of the minus sign in (2.7). One could consider making the transformation $x_3 \leftrightarrow x_4$, $x_1 \leftrightarrow x_2/\lambda$, $\omega \to -\omega$ then we would recover the same quadratic Hamiltonian (2.6) and linearised ODE (2.7). Hence the sign of ω would appear to be irrelevant. This is not the case, however, since such a transformation would change the action of the reversibility (2.4). So if we have chosen co-ordinates so that x_2 and x_4 are the momentum variables, the sign of the product of the coefficients in front of these variables, $-\omega$, is an important geometrical property and is called the *Birkhoff signature* (see e.g. [42,13] for some of its consequences).

We shall be interested in what follows in the singular limit $\lambda/\omega \to 0$. For definiteness, the analysis is carried out by assuming that $\omega = O(1)$ but that $\lambda \to 0$. As already discussed, the 'fast radiation' limit that arises in many examples actually has $\omega \to \infty$ while $\lambda = O(1)$, but it is perfectly valid to interpret our results in that way, the important ingredient being the ratio λ/ω.

Within the space of reversible, fourth-order Hamiltonian systems, an equilibrium undergoing a singular saddle-centre bifurcation — that is where $\alpha = \lambda^2$ passing through zero in (2.6) — is of codimension one (see Fig. 1). Our analysis shall only concern positive α, so we henceforth continue to use λ as a real parameter. It has become quite standard to simplify a dynamical system near such a local bifurcation by using normal form analysis, see for example

[20,18]. Such a normal form in the case of interest here will be essentially determined by the linear part of f, the reversibility and the assumed Hamiltonian structure. The procedure is simple. First one calculates the normal form near a saddle-centre equilibrium without taking the extra structure. Then further restrictions on the normal form are imposed by assuming that it inherits the structure of the original vector field. Hence the normal form in the vicinity of the type-I singular saddle-centre point can be written as

$$\begin{aligned}
\dot{x_1} &= x_2, \\
\dot{x_2} &= \lambda^2 x_1 + \frac{\partial P}{\partial x_1}(x_1, K; \lambda, \omega, \mu), \\
\dot{x_3} &= -x_4 \left[\omega + 2\frac{\partial P}{\partial K}(x_1, K; \lambda, \omega, \mu)\right], \\
\dot{x_4} &= x_3 \left[\omega + 2\frac{\partial P}{\partial K}(x_1, K; \lambda, \omega, \mu)\right],
\end{aligned}$$

where

$$K = x_3^2 + x_4^2$$

and P is an arbitrary polynomial in its first two arguments, whose coefficients in general depend on λ, ω and any other parameters μ present in the problem. The corresponding Hamiltonian is

$$H^{nf} = H^{lin} + P(x_1, x_3^2 + x_4^2; \lambda, \omega, \mu).$$

The polynomial P may be truncated at any order to produce a reduced-order normal form. Note that a convenient way of performing the computations of P up to any order is to work with the Hamiltonian function itself, see e.g. [40]. At whichever order we truncate, the normal form (2.8) is completely integrable with invariants H^{nf} and K. Note therefore, that although we may obtain better accuracy to solutions to the original ODE in a neighbourhood of a saddle centre *over a finite time interval* by taking higher-order truncations, we will never obtain the true qualitative phase portraits unless the underlying ODE were itself completely integrable. The phenomena of concern in this paper specifically arise through non-integrability as ESs do not exist in integrable systems. Therefore we shall explicitly have to consider perturbations which break the normal form structure if we want to capture the true dynamics.

We shall also draw a distinction in what follows between ODEs that in addition have odd symmetry (like the cubic NLS system in Section 4.2 below) or have odd symmetry in one of their components (like the equations for SHG in Section 4.3). The simplest unperturbed normal forms we study are those where P is truncated at the lowest order nonlinear terms commensurate with the symmetry. Truncation to a quadratic normal form corresponds to the

Hamiltonian

$$H_1^{nf} = \frac{1}{2}(x_2^2 - \lambda^2 x_1^2) - \frac{\omega}{2}(x_3^2 + x_4^2) + \frac{1}{3}a x_1^3 + b x_1(x_3^2 + x_4^2), \qquad (2.9)$$

which is invariant under the transformation R given by (2.4). Similarly, truncation to a cubic normal form corresponds to the Hamiltonian

$$H_2^{nf} = \frac{1}{2}(x_2^2 - \lambda^2 x_1^2) - \frac{\omega}{2}(x_3^2 + x_4^2) + \frac{1}{4}a x_1^4 + \frac{1}{2}b x_1^2(x_3^2 + x_4^2) - \frac{1}{4}c(x_3^2 + x_4^2)^2, \quad (2.10)$$

which is invariant under R and also under odd symmetry

$$S : (x_1, x_2, x_3, x_4) \to -(x_1, x_2, x_3, x_4). \qquad (2.11)$$

Here a, b and c are arbitrary constants (functions of λ, ω and μ in general) which must be evaluated for specific examples.

To these truncated Hamiltonians, we will consider the effects of adding normal-form breaking terms (that is nonlinear terms which break the invariance of K). In the next section, where we consider a careful asymptotic analysis of what happens to the sech^2 solution of the ODE system associated with (2.9) (or the sech solution to that associated with (2.10)), we shall worry about the relative size of these perturbation terms as $\lambda \to 0$. However, for the time being we shall consider only those nonlinear terms that are of the lowest order in x. In fact, arguments in [37] show that the question of the size of the minimum tail amplitude of the QSs found by perturbing the sech^2-solution is determined solely by perturbation terms of the form

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^{n_1} x_2^{m_1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \\ x_1^{n_2} x_2^{m_2} \\ 0 \end{pmatrix}, \qquad n_i + m_i \geq 2, \qquad (2.12)$$

added to the the truncated normal form. To conserve the overall $R-$reversibility of the truncated form, we must furthermore restrict ourselves to the cases when m_1 is even and m_2 is odd. Hence, for quadratic perturbations to the normal form, the following terms added to the Hamiltonian are the only ones that need be considered:

$$x_1^2 x_3, \quad x_2^2 x_3, \quad x_1 x_2 x_4. \qquad (2.13)$$

Similarly, for cubic perturbations to the normal form, we only need consider those resulting from the following terms in the Hamiltonian:

$$x_1^3 x_3, \quad x_1 x_2^2 x_3, \quad x_1^2 x_2 x_4, \quad x_2^3 x_4. \qquad (2.14)$$

These perturbations are purely odd-symmetric, i.e. invariant under S. Note that invariance under R of the Hamiltonian implies a reversibility (equivariance under additional reversal of time) for the corresponding ODEs, whereas invariance under S of the Hamiltonian implies a $\mathbb{Z}_2$ symmetry of the ODEs, and an additional reversibility SR.

3 Asymptotic analysis

We shall now use the perturbed normal forms to perform an asymptotic analysis of the existence of BS in the limit $\lambda \to 0$. There are essentially two steps in the analysis, the first is to write a condition for zero tail asymptotics as we allow the separation between the bound states to vary. The second is to use the approach of [22] view the BS as a superposition of weakly interacting particle-like objects in the PDE Hamiltonian and to demand that their interaction vanishes, thus formally finding a steady solution to the PDE. First though, as a starting point of the analysis, we need an expression for the QS solutions to the appropriately perturbed truncated normal forms. We shall treat three cases separately, depending on the symmetry properties of the underlying ODEs, by taking perturbations of the form (2.13) and (2.14) to either (2.9) or (2.10).

3.1 Systems without $\mathbb{Z}_2$ symmetry

Systems which are not invariant under any additional symmetry other than the reversibility R, generically have lowest-order truncated normal forms which may be derived from H_1^{nf}

$$
\begin{aligned}
\dot{x_1} &= x_2, \\
\dot{x_2} &= \lambda^2 x_1 - a x_1^2 - b(x_3^2 + x_4^2), \\
\dot{x_3} &= -(\omega - 2bx_1)x_4, \\
\dot{x_4} &= (\omega - 2bx_1)x_3.
\end{aligned}
\tag{3.1}
$$

The homoclinic-to-zero solution of (3.1) exists for any $a \neq 0$ and can be written explicitly as

$$
\begin{aligned}
x_1 &= \frac{3}{2a}\lambda^2 \operatorname{sech}^2(\lambda t/2), \\
x_2 &= -\frac{3}{2a}\lambda^3 \tanh(\lambda t/2) \operatorname{sech}^2(\lambda t/2), \\
x_3 &= 0, \quad x_4 = 0.
\end{aligned}
\tag{3.2}
$$

A one-parameter family of periodic orbits (up to phase shift) can also be written explicitly as

$$
\begin{aligned}
x_1 &= T(K) := \frac{3\lambda^2 + \sqrt{9\lambda^4 - 48abK^2}}{4a},\\
x_2 &= 0,\\
x_3 &= K\cos[\omega^*(K)(t+\tau)],\\
x_4 &= K\sin[\omega^*(K)(t+\tau)],
\end{aligned}
\tag{3.3}
$$

where $\omega^*(K) = \omega - 2bT(K)$, and $0 < K < K_{max} = O(\lambda^2)$ (see [37]).

3.1.1 *Zeroth step; single humped quasi-soliton*

We now proceed in writing down an expression for the single-humped QS solutions that arise from perturbations of the form (2.13) or (2.14) added to the truncated normal form H_1^{nf}. This step is non-rigorous in the pre-exponent factor and a more detailed calculation requires careful beyond-all-orders asymptotics that is beyond the scope of an analysis via normal form.

In the singular saddle-centre limit quasi-solitons are characterized by exponentially small amplitude of radiation. This *a priori* knowledge and expression (3.2) for the unperturbed soliton allows us to assume the following scaling,

$$
x_1 = O(\lambda^2), \quad x_2 = O(\lambda^3), \quad x_3 = O(e^{-\kappa/\lambda}), \quad x_4 = O(e^{-\kappa/\lambda}), \tag{3.4}
$$

for some positive constant κ that is independent of λ. From these estimates, we find that the lowest order perturbation among the set (2.13) and (2.14) is proportional to $x_1^2 x_3$. Hence we take a perturbed Hamiltonian in the form

$$
H_1^{pert} = H_1^{nf} + \alpha_1 x_1^2 x_3 + O(\lambda^5 e^{-\kappa/\lambda}), \tag{3.5}
$$

where α_1 is an arbitrary constant and H_1^{nf} is given by expression (2.9). The Hamiltonian (3.5) defines the following dynamical equations

$$
\begin{aligned}
\ddot{x}_1 - \lambda^2 x_1 + a x_1^2 &= -b(x_3^2 + x_4^2) - 2\alpha_1 x_1 x_3,\\
\ddot{x}_3 + \tilde{\omega}^2 x_3 &= \tilde{\omega}\alpha_1 x_1^2,
\end{aligned}
\tag{3.6}
$$

where $\tilde{\omega} = \omega - 2bx_1 = \omega + O(\lambda^2)$. To the leading-order, the estimates (3.4) allow us to neglect the right-hand-side of the first equation in system (3.6) and replace $\tilde{\omega}$ by ω in the second equation. Solutions to the resulting system take the form

$$
\begin{aligned}
x_1(t) &= \frac{3}{2a}\,\lambda^2\,\mathrm{sech}^2(\lambda t/2),\\
x_3(t) &= A\cos\omega t + B\sin\omega t + \qquad\qquad\qquad\qquad\qquad\qquad (3.7)\\
&\quad +\sin\omega t\int_{-\infty}^{t}\alpha_1 x_1^2(t')\cos\omega t'\,dt' - \cos\omega t\int_{-\infty}^{t}\alpha_1 x_1^2(t')\sin\omega t'\,dt,
\end{aligned}
$$

where A, B are unknown constants. We are interested in the R-symmetric solution to (3.7). Hence, imposing the condition that x_1 and x_2 must be even, we see that asymptotics of the solution in the far field must satisfy

$$
x_3(t) = \frac{B}{\sin\phi}\cos(\omega t \pm \phi), \quad t \to \pm\infty, \tag{3.8}
$$

where $2\phi = \arcsin\dfrac{2B}{\sqrt{A^2+B^2}}$ is the *phase shift* between the periodic solutions at $t = \pm\infty$. This formula, once an expression is found for the constant B, provides a leading order relation between the amplitude of the tail radiation and the phase shift. Note that the amplitude becomes singular when $\phi = m\pi$, for any integer m.

Now, it is the estimate of B that requires in general beyond all orders asymptotics. Proceeding formally at this stage, assuming (3.7) to be exact rather than a leading-order expression, we find that amplitude of the radiation is proportional to the Fourier spectrum of the force acting on the oscillator in Eqs. (3.6),

$$
B = -\frac{1}{2}\int_{-\infty}^{+\infty}\alpha_1 x_1^2(t')\cos\omega t'\,dt'. \tag{3.9}
$$

Direct calculation of the integral (3.9) gives

$$
B = -\frac{3\pi\alpha_1\omega(\omega^2+\lambda^2)}{a^2}\,\mathrm{csch}\,\frac{\pi\omega}{\lambda} = -\frac{3\pi\omega^3}{a\sqrt{|\omega|}}\,Ce^{-\pi|\omega|/\lambda}\left[1+O(\lambda^2)\right], \tag{3.10}
$$

with

$$
C = C_{est} = \frac{2\alpha_1}{a}\sqrt{|\omega|}. \tag{3.11}
$$

In fact, the work of Lombardi [37] shows that a general expression for the tail amplitude can only be obtained after considering *all* perturbation terms to the truncated normal form. Nevertheless, the resulting formula is of the form (3.10), in particular it has the same exponent. But the evaluation of C is highly problem dependent as it involves the entire nonlinearity of any example system. Techniques for estimating C using exponential asymptotics were mentioned in the Introduction, but are beyond the scope of this paper.

The result then is that we assume the QS pulse to be given to leading order by

$$
\begin{aligned}
x_1 &= \frac{3}{2a}\lambda^2 \mathrm{sech}^2(\lambda t/2), \\
x_2 &= -\frac{3}{2a}\lambda^3 \tanh(\lambda t/2)\mathrm{sech}^2(\lambda t/2), \\
x_3 &= \frac{B}{\sin\phi}\cos[\omega(t+\phi\tanh t/2)], \\
x_4 &= \frac{B}{\sin\phi}\sin[\omega(t+\phi\tanh t/2)],
\end{aligned}
\tag{3.12}
$$

where B is given by (3.10) for some factor C (note the tanh-function representation of the phase shift arises in the more precise calculations of Lombardi and is valid for sufficiently small ϕ).

3.1.2 First step; zero tail radiation

We construct a two-pulse BS within the method by superimposing two copies of (3.12) end-to-end a distance $\Delta t \gg 1$ apart. Assuming that the Hamiltonian (3.5) has no $\mathbb{Z}_2$-symmetry implies that only a superposition that results in an R-symmetric solution need be considered (a simple argument shows that non-symmetric homoclinic solutions are of higher-codimension). Shifting time so that $t = t_0 = -\Delta t/2$ rather than $t = 0$ is the centre of symmetry, we therefore look for BS in the form

$$
\begin{aligned}
x_1^{bs}(t) &= x_1(t) + x_1(t+\Delta t), \\
x_3^{bs}(t) &= x_3(t) + x_3(t+\Delta t).
\end{aligned}
\tag{3.13}
$$

The requirement that the solution is a true homoclinic solution rather than a QS, i.e. that it asymptotes to zero as $t \to \pm\infty$ yields the 'trapping conditions'

$$
\cos[\omega t \pm \phi] + \cos[\omega(t+\Delta t) \pm \phi] = 0. \tag{3.14}
$$

The solution to (3.14) is independent of ϕ:

$$
\omega\Delta t = (2n-1)\pi, \quad n = 1, 2, 3 \ldots. \tag{3.15}
$$

The value of the positive integer n determines the order of the standing wave separating the two sech^2 cores. For a symmetric BS, $n = (n_{ext}+1)/2$, where n_{ext} is the number of extrema of the standing wave, see Fig. 2.

3.1.3 Second step; stationarity of interaction

Recall the relation (2.5) we assume between the Hamiltonian $\mathcal{H}$ of the evolution PDE and the Hamiltonian H of the travelling-wave ODE. Now we use the

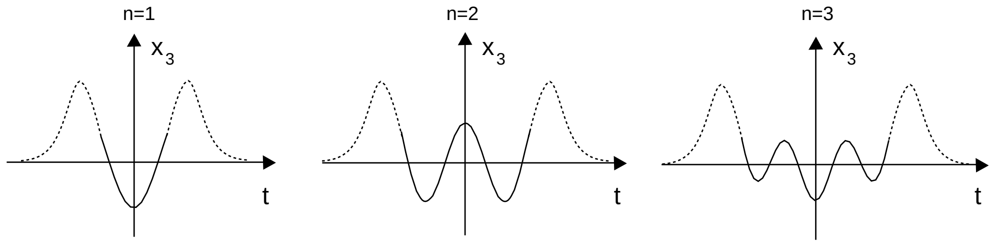

Fig. 2. Qualitative examples of symmetric bound states for three successive orders n. Standing waves are depicted as solid lines, and the quasi-soliton cores as dashed lines.

asymptotic approach of Gorshkov and Ostrovsky where we assume that the BS (3.13) consists of two interacting particles whose evolution is described by the (2.5). To be stationary, the BS must realize an extremum of the interacting potential U_{int} which is the non-quadratic part of $\mathcal{H}$ and hence, by assumption (2.5), the non-quadratic part of the Hamiltonian (3.5):

$$U_{int} = k \int_{-\infty}^{+\infty} \left[\frac{1}{3} a (x_1^{bs})^3 + \alpha_1 (x_1^{bs})^2 x_3^{bs}\right] dt + O(\lambda^5 e^{-\kappa/\lambda}). \tag{3.16}$$

In contrast to the standard implementation of the method, we consider a modification (proposed in [7]) in which the 'particles' we consider consist of a 'core' [the (x_1, x_2) sech^2 component] and 'tail' [the (x_3, x_4) exponentially small periodic part]. Substitution of expressions (3.13) into the integral (3.16) results in an expression for the interaction potential that is composed of the core-core interaction U_{c-c} and the core-tail interaction U_{c-t} , where

$$U_{c-c}(\Delta t, \phi) = ka \int_{-\infty}^{+\infty} \left[x_1^2(t) x_1(t+\Delta t) + x_1(t) x_1^2(t+\Delta t)\right] dt, \tag{3.17}$$

$$\begin{aligned} U_{c-t}(\Delta t, \phi) = k\alpha_1 \int_{-\infty}^{+\infty} \left[x_1^2(t) x_3(t+\Delta t) + x_1^2(t+\Delta t) x_3(t)\right] dt + \\ +2k\alpha_1 \int_{-\infty}^{+\infty} x_1(t) x_1(t+\Delta t) \left[x_3(t) + x_3(t+\Delta t)\right] dt. \end{aligned} \tag{3.18}$$

Now, after substitution of x from (3.12) we can proceed to calculate the leading order terms in integrals (3.17) and (3.18). Calculation of U_{c-c} and the first integral in the expression for U_{c-t} is straightforward. An estimate for the second integral in (3.18) can be obtained using the Cauchy-Buniakowski-Schwartz inequality,

$$\begin{aligned} &\left|\int_{-\infty}^{+\infty} x_1(t) x_1(t+\Delta t) x_3(t)\, dt\right| \le \\ &\le \sqrt{\int_{-\infty}^{+\infty} x_1^2(t) x_1^2(t+\Delta t)\, dt} \sqrt{\int_{-\infty}^{+\infty} x_3^2(t)\, dt} = O\left[\lambda^{7/2} (\lambda \Delta t)^{1/2} e^{-\lambda \Delta t - \pi|\omega|/\lambda}\right]. \end{aligned} \tag{3.19}$$

As a result, the integral (3.19) contributes to the higher-order terms if λ is small and the QS cores are sufficiently far separated so that

$$\Delta t > \frac{\pi|\omega|}{\lambda^2}.$$

(Note that this assumption is consistent with the asymptotic solution (3.23) below).

The final expression for the interaction potential is given by

$$\begin{aligned} U_{int}(\Delta t, \phi) &= U_{c-c}(\Delta t, \phi) + U_{c-t}(\Delta t, \phi) + O\left(\lambda^5 e^{-\kappa/\lambda}\right) = \\ &= 4k\left[\frac{36\lambda^5}{a^2}\exp(-\lambda\Delta t) - \frac{B^2}{\sin\phi}\cos(\omega\Delta t + \phi)\right] + O\left(\lambda^5 e^{-\kappa/\lambda}\right). \end{aligned} \tag{3.20}$$

The first term in each sum in Eqs. (3.20) comes from the core-core interaction and the second from the core-tail. Note the physical interpretation of this expression. The core-core interaction is providing a repulsion (for $k > 0$) that decays exponentially with Δt, whereas the core-tail is providing an attraction or repulsion that depends periodically on Δt. Hence as Δt varies, provided it is big enough we should expect to find a series of extrema of (3.20). Within the method, these extrema correspond to the BSs that we seek.

Extrema of the interaction potential are defined by the following conditions

$$\begin{aligned} \frac{\partial U_{int}}{\partial \phi} &\equiv 4kB^2\frac{\sin\omega\Delta t}{\sin^2\phi} = 0, \\ \frac{\partial U_{int}}{\partial(\Delta t)} &\equiv 4k\left[-\frac{36\lambda^6}{a^2}\exp(-\lambda\Delta t) + \frac{\omega B^2}{\sin\phi}\sin(\omega\Delta t + \phi)\right] = 0. \end{aligned} \tag{3.21}$$

These equation can be simplified by using the trapping condition (3.15), which we consider as specifying Δt in terms of ω and n. Then the first condition in Eqs. (3.21) is always satisfied. The second condition for the extrema, after substitution for B from (3.10) leads to the following requirement,

$$\frac{4}{\pi^2 C^2}\left(\frac{\lambda}{\omega}\right)^6 e^{-(2n-1)\pi\lambda/|\omega|} + \frac{\omega}{|\omega|}e^{-2\pi|\omega|/\lambda} = 0. \tag{3.22}$$

The first term in Eq. (3.22) is strictly positive. The sign of the second term is $\mathrm{sign}(\omega)$ which is the negative of the Birkhoff signature. Therefore *in order to find any solutions we require that the Birkhoff signature be positive* (note that is precisely the same condition as was obtained in [42] when considering BS accumulating on a regular ES, rather than the singular limit). If this signature condition is satisfied, then bound states of symmetric QSs are given by those values of ω/λ for which equality can be found for some n. Assuming $\omega < 0$, we can rearrange (3.22) to write it in the form

$$n(\Omega) = \Omega^2 - \frac{3}{\pi}\Omega \log \Omega - \frac{1}{\pi}\Omega \log \frac{\pi C}{2} + \frac{1}{2}, \quad \text{where} \quad \Omega = \frac{-\omega}{\lambda}, \qquad \text{for } \omega < 0. \tag{3.23}$$

Note that the first two terms

$$n \sim \Omega_n^2 - \frac{3}{\pi}\Omega_n \log \Omega_n + O(\Omega_n) \ \text{ as } \ n \to \infty$$

are entirely universal and are independent of the beyond-all-orders calculation of C.

3.2 Systems with odd symmetry

Now suppose that our reversible Hamiltonian system (2.2), (2.3) is additionally invariant under the simple action of $\mathbb{Z}_2$

$$S : x \to -x.$$

Perturbed normal forms for such systems can be constructed by perturbation of the Hamiltonian H_2^{nf} with the quartic terms (2.14), since the symmetry precludes any cubic terms in the Hamiltonian. We shall repeat the above calculation for this special case.

We start with the zeroth step. The lowest-order truncated normal form is calculated from (2.10):

$$\begin{aligned}
\dot{x_1} &= x_2, \\
\dot{x_2} &= \lambda^2 x_1 - a x_1^3 - b x_1 (x_3^2 + x_4^2), \\
\dot{x_3} &= -[\omega - 2b x_1^2 + c(x_3^2 + x_4^2)] x_4, \\
\dot{x_4} &= [\omega - 2b x_1^2 + c(x_3^2 + x_4^2)] x_3.
\end{aligned} \tag{3.24}$$

Solutions which are homoclinic to zero exist when the parameter $a > 0$. Under this requirement, the soliton is sech rather than sech^2:

$$\begin{aligned}
x_1 &= \sqrt{\frac{2}{a}}\,\lambda\,\mathrm{sech}(\lambda t), \\
x_2 &= -\sqrt{\frac{2}{a}}\,\lambda^2\ \tanh(\lambda t)\,\mathrm{sech}(\lambda t), \\
x_3 &= 0, \quad x_4 = 0.
\end{aligned} \tag{3.25}$$

The one-parameter family of periodic orbits are now given by

$$\begin{aligned} x_1 &= T(K) := \sqrt{2\frac{\lambda^2 - bK^2}{a}}, \\ x_2 &= 0, \\ x_3 &= K\cos[\omega^*(K)(t+\tau)], \\ x_4 &= K\sin[\omega^*(K)(t+\tau)], \end{aligned} \tag{3.26}$$

where τ is a phase shift, $\omega^*(K) = \omega - 2bT(K) + cK^2$ and $0 < K < K_{max} = O(\lambda)$.

Considering cubic perturbations (2.14), and seeking quasi-solitons with exponentially small tail amplitude, we assume that

$$x_1 = O(\lambda), \quad x_2 = O(\lambda^2), \quad x_3 = O(e^{-\kappa/\lambda}), \quad x_4 = O(e^{-\kappa/\lambda}), \tag{3.27}$$

for some λ-independent $\kappa > 0$. From this we deduce the lowest-order term from the of possible perturbations (2.14), and so assume a perturbed Hamiltonian of the form

$$H_2^{pert} = H_2^{nf} + \alpha_1 x_1^3 x_3 + O(\lambda^4 e^{-\kappa/\lambda}), \tag{3.28}$$

where $a > 0$ and α_1, b and c are arbitrary constants to be determined for particular examples. The resulting equations of motion are

$$\begin{aligned} \ddot{x}_1 - \lambda^2 x_1 + a x_1^3 &= -b x_1 (x_3^2 + x_4^2) - 3\alpha_1 x_1^2 x_3, \\ \ddot{x}_3 + \tilde{\omega}^2 x_3 &= \tilde{\omega}\alpha_1 x_1^3, \end{aligned} \tag{3.29}$$

where $\tilde{\omega} = \omega - b x_1^2 + c(x_3^2 + x_4^2) = \omega + O(\lambda^2)$. Again, estimates (3.27) allows us to neglect the right-hand side in the first equation in (3.29) and replace $\tilde{\omega}$ with ω in the second one.

To leading order, the R-symmetric solution to Eqs. (3.29) consists of the core (3.25) and has the same asymptotics in far field as in (3.8). The amplitude of radiation is however now given by

$$B = -\frac{1}{2}\int_{-\infty}^{+\infty} \alpha_1 x_1^3(t') \cos\omega t' \, dt'. \tag{3.30}$$

Direct calculation of the integral (3.30) for the homoclinic solution (3.25) leads to the following expression

$$B = -\frac{\pi\alpha_1(\omega^2 + \lambda^2)}{a\sqrt{2a}} \operatorname{sech}\frac{\pi\omega}{2\lambda} = -\frac{\sqrt{2}\pi\omega^2}{\sqrt{a|\omega|}} C\, e^{-\pi|\omega|/(2\lambda)}[1 + O(\lambda^2)], \tag{3.31}$$

where an estimate for factor C is given by

$$C = C_{est} = \frac{\alpha_1}{a}\sqrt{|\omega|}. \qquad (3.32)$$

Similar comments apply about how this estimate is in fact only qualitative and one should instead consider C as a model-specific factor. So we take, as the output of the zeroth step that the QS is given by

$$\begin{aligned}
x_1 &= \sqrt{\frac{2}{a}}\,\lambda\,\mathrm{sech}(\lambda t), \\
x_2 &= -\sqrt{\frac{2}{a}}\,\lambda^2\,\tanh(\lambda t)\,\mathrm{sech}(\lambda t), \\
x_3 &= \frac{B}{\sin\phi}\cos[\omega(t+\phi\tanh t/2)], \\
x_4 &= \frac{B}{\sin\phi}\sin[\omega(t+\phi\tanh t/2)],
\end{aligned} \qquad (3.33)$$

where B is given by (3.31) for some constant C.

In the first step, as before, we construct a BS as a two-pulse solution

$$\begin{aligned}
x_1^{bs}(t) &= x_1(t) + r x_1(t+\Delta t), \\
x_3^{bs}(t) &= x_3(t) + r x_3(t+\Delta t).
\end{aligned} \qquad (3.34)$$

Here $r = \pm 1$ and reflects the fact that pulses can be formed that are symmetric under either R or SR, that is the profiles $x_{1,3}(t)$ are either both *even* or *odd*. The trapping condition now takes the form

$$\omega\Delta t = \left(2n - \frac{1+r}{2}\right)\pi, \quad n = 1, 2, 3\ldots \qquad (3.35)$$

Again, the value of the positive integer n determines the order of the standing wave separating the two sech cores. If n_{ext} is the number of extrema of the standing wave, for a symmetric BS $n = (n_{ext}+1)/2$ (see Fig. 2) whereas for an anti-symmetric BS $n = n_{ext}/2$ (see Fig. 3).

In the second step, the existence of a bound state is determined by the interaction potential determined from the non-quadratic part of the Hamiltonian (3.28),

$$U_{int} = k\int_{-\infty}^{+\infty}\left[\frac{1}{4}a(x_1^{bs})^4 + \alpha_1(x_1^{bs})^3 x_3^{bs}\right]dt + O(\lambda^4 e^{-\kappa/\lambda}). \qquad (3.36)$$

The core-core interaction U_{c-c} and the core-tail interaction U_{c-t} are here given

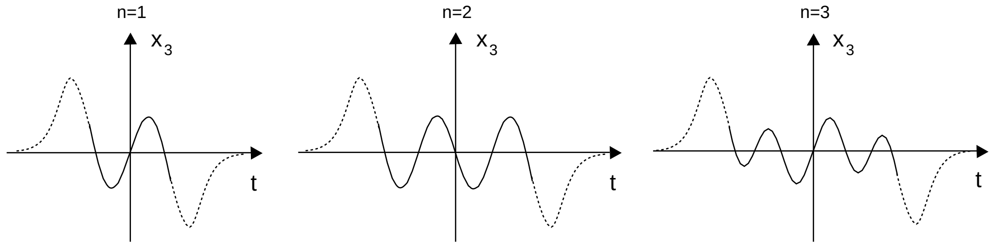

Fig. 3. Examples of anti-symmetric bound states of three successive orders, depicted qualitatively as in Fig. 2.

by

$$U_{c-c}(\Delta t,\phi) = rka \int_{-\infty}^{+\infty} \left[x_1^3(t)x_1(t+\Delta t) + x_1(t)x_1^3(t+\Delta t)\right]\, dt + \\ +\frac{3}{2}ka \int_{-\infty}^{+\infty} x_1^2(t)x_1^2(t+\Delta t)\, dt, \tag{3.37}$$

$$U_{c-t}(\Delta t,\phi) = rk\alpha_1 \int_{-\infty}^{+\infty} \left[x_1^3(t)x_3(t+\Delta t) + x_1^3(t)x_3(t+\Delta t)\right]\, dt + \\ +3k\alpha_1 \int_{-\infty}^{+\infty} x_1^2(t)x_1(t+\Delta t)\left[rx_3(t) + x_3(t+\Delta t)\right]\, dt + \\ +3k\alpha_1 \int_{-\infty}^{+\infty} x_1(t)x_1^2(t+\Delta t)\left[x_3(t) + rx_3(t+\Delta t)\right]\, dt. \tag{3.38}$$

Substitution of expression (3.33) for the quasi-soliton into integrals (3.37) and (3.38) makes evaluation of the potential for core-core interaction and the first term of the core-tail interaction straightforward. Using the Cauchy-Buniakowski-Schwartz inequality we can estimate the last two integral in the expression for U_{c-t},

$$\left|\int_{-\infty}^{+\infty} x_1^2(t)x_1(t+\Delta t)x_3(t)\, dt\right| \leq \\ \leq \sqrt{\int_{-\infty}^{+\infty} x_1^4(t)x_1^2(t+\Delta t)\, dt}\sqrt{\int_{-\infty}^{+\infty} x_3^2(t)\, dt} = O(\lambda^{5/2}e^{-\lambda\Delta t - \pi|\omega|/(2\lambda)}). \tag{3.39}$$

The integral (3.39) contributes to the higher-order terms if the separation of quasi-solitons satisfy

$$\Delta t > \frac{\pi|\omega|}{2\lambda^2} \tag{3.40}$$

(which is borne out by the asymptotic solution below). Under this assumption, the interaction potential is given by

$$U_{int}(\Delta t,\phi) = U_{c-c}(\Delta t,\phi) + U_{c-t}(\Delta t,\phi) = \\ = 4kr\left[\frac{8\lambda^3}{a}e^{-\lambda\Delta t} - \frac{B^2}{\sin\phi}\cos(\omega\Delta t + \phi)\right] + O(\lambda^4\, e^{-\kappa/\lambda}). \tag{3.41}$$

After taking into account (3.31) and the trapping condition (3.35), calculation of the extremum of the interaction potential leads to a criterion for existence of BS analogous to (3.22):

$$\frac{4}{\pi^2 C^2}\left(\frac{\lambda}{\omega}\right)^4 e^{-(4n-1-r)\pi\lambda/|\omega|/2} + \frac{r\omega}{|\omega|}\, e^{-\pi|\omega|/\lambda} = 0. \qquad (3.42)$$

The criterion (3.42) incorporates the product of the Birkhoff signature and r which describes the symmetry assumed of the BSs. Specifically, we require $r\omega < 0$. So now, no matter what the Birkhoff signature is, we find BSs — even solutions if the signature is negative, and odd solutions if it is positive. Rearranging the formula to write it as $n(\Omega)$ where $\Omega = |\omega|/\lambda$, one obtains

$$n(\Omega) = \frac{1}{2}\Omega_n^2 - \frac{2}{\pi}\Omega_n \ln\, \Omega_n - \frac{1}{\pi}\Omega_n \ln\frac{\pi C}{2} + \frac{1+r}{4}, \quad \text{if } r\omega < 0. \qquad (3.43)$$

and the leading order terms (in both cases $r = \pm 1$) is given by

$$n \sim \frac{1}{2}\Omega_n^2 - \frac{2}{\pi}\Omega_n \log \Omega_n + O(\Omega_n) \ \text{ as } n \to \infty.$$

Note that this asymptotic expression agrees with that for the NLS equation perturbed with a third-order dispersive term in [35,9] for which the Birkhoff signature is positive and hence even (R-symmetric) multi-pulse solutions occur.

3.3 Systems with mixed symmetry

In the previous sections we have considered dynamical systems whose solutions are purely even or odd. However, many physically meaningful two-component systems (for example, the second harmonic generation model in Section 4.3 below) have solutions with mixed symmetry, i.e. that are even in one component and odd in the other. In this section we consider the case when the system (2.2), (2.3) is additionally invariant under the action of $\mathbb{Z}_2$,

$$S_3 : (x_1, x_2, x_3, x_4) \to (x_1, x_2, -x_3, -x_4), \qquad (3.44)$$

so that the x_1 component is always even, but the x_3-component may be even or odd.

Within the method adopted in this work, systems with this symmetry appear as a result of perturbation of the Hamiltonian (2.9) by quadratic terms (2.13). Moreover, the scaling (3.27) allows us to write the lowest-order perturbed Hamiltonian in the form

$$H_3^{pert} = H_2^{nf} + \alpha_1 x_1^2 x_3 + O(\lambda^4 e^{-\kappa/\lambda}), \qquad (3.45)$$

where α_1 is some constant.

The Hamiltonian (3.45) defines equations of motion which to leading order have a symmetric QS solution given by

$$\begin{aligned} x_1(t) &= \sqrt{\frac{2}{a}}\,\lambda\,\mathrm{sech}(\lambda t), \\ x_3 &= \frac{B}{\sin\phi}\cos[\omega(t+\phi\tanh t/2)], \end{aligned} \tag{3.46}$$

where amplitude of the radiation

$$B = -\frac{1}{2}\int_{-\infty}^{+\infty} \alpha_1 x_1^2(t')\cos\omega t'\,dt'. \tag{3.47}$$

Direct calculation of the integral (3.47) yields

$$B = -\frac{\pi\alpha_1\omega}{a}\,\mathrm{csch}\frac{\pi\omega}{2\lambda} = -\frac{\sqrt{2}\pi\omega^2}{\sqrt{a|\omega|}}\,C\,e^{-\pi|\omega|/(2\lambda)}[1+O(e^{-\pi|\omega|/(2\lambda)})], \tag{3.48}$$

where an estimate based on the lowest-order term only for the constant C is given by

$$C = C_{est} = \frac{\sqrt{2}\alpha_1}{\sqrt{a}\omega}\sqrt{|\omega|}. \tag{3.49}$$

According to symmetry of the Hamiltonian (3.45) we construct a BS as a two-pulse solution

$$\begin{aligned} x_1^{bs}(t) &= x_1(t) + r x_1(t+\Delta t), \\ x_3^{bs}(t) &= x_3(t) + x_3(t+\Delta t), \end{aligned} \tag{3.50}$$

where $r = \pm 1$. The trapping condition takes the same form as (3.15)

$$\omega\Delta t = (2n-1)\pi, \quad n = 1, 2, 3\ldots \tag{3.51}$$

If the separation Δt of the quasi-solitons in a BS are related by

$$\Delta t > \frac{\pi|\omega|}{2\lambda^2}.$$

then the leading order expression for the interaction potential is

$$U_{int}(\Delta t,\phi) = 4k\left[\frac{8r}{a}\lambda^3\exp(-\lambda\Delta t) - \frac{B^2}{\sin\phi}\cos(\omega\Delta t+\phi)\right] + O(\lambda^4 e^{-\kappa/\lambda}). \tag{3.52}$$

After taking into account the trapping condition (3.15) and the amplitude of the tail (3.48), extrema of the interaction potential (3.52) satisfy

$$\frac{4}{\pi^2 C^2}\left(\frac{\lambda}{\omega}\right)^4 e^{-(2n-1)\pi\lambda/|\omega|} + \frac{r\omega}{|\omega|}\, e^{-\pi|\omega|/\lambda} = 0. \tag{3.53}$$

Similarly to the case of systems with pure odd symmetry, existence of BS is governed by the condition $r\omega < 0$ implying that BSs exist regardless of Birkhoff signature. Again, rearranging the formula to write it as $n(\Omega)$ where $\Omega = |\omega|/\lambda$, one obtains

$$n(\Omega) = \frac{1}{2}\Omega_n^2 - \frac{2}{\pi}\Omega_n \ln \Omega_n - \frac{1}{\pi}\Omega_n \ln\frac{\pi C}{2} + \frac{1}{2}, \quad \text{if } r\omega < 0, \tag{3.54}$$

which simplifies to

$$n \sim \frac{1}{2}\Omega_n^2 - \frac{2}{\pi}\Omega_n \log\Omega_n + O(\Omega_n) \ \text{ as } \ n \to \infty.$$

4 Examples

4.1 A coupled KdV system

Grimshaw and Cook [24] showed using asymptotics that a certain coupled system of two coupled KdV equations posses fundamental ES solutions. There are various more or less equivalent forms of such systems that arise in the study of strongly interacting nonlinear waves, for example motivated by internal fluid waves, planetary waves and harmonic generation in optics (see [12] and references therein). Here we shall take the following general form where all the coupling is through the nonlinearity

$$\begin{aligned} u_\tau + u_{xxx} + s(uv)_x + r(v^2)_x + 2q_1 uu_x &= 0,\\ v_\tau + v_{xxx} + r(uv)_x + s(u^2)_x + 2q_2 uu_x + \eta v_x &= 0, \end{aligned} \tag{4.1}$$

and show that it possesses BS solutions of the kind described by the above theory.

On making a reduction to a steady frame via $t = x - c\tau$, and integrating once with respect to t, we obtain the system of ODEs

$$\begin{aligned} u'' &= \lambda^2 u - suv - \frac{r}{2}v^2 - q_1 u^2,\\ v'' &= -\omega^2 v - ruv - \frac{s}{2}u^2 - q_2 v^2, \end{aligned} \tag{4.2}$$

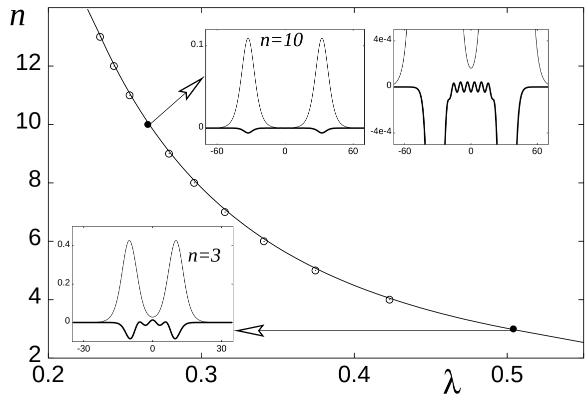

Fig. 4. Comparison between theory and numerics in the model with coupled KdV equations. Dots correspond to the numerical results and the line to the formulae (3.23), which is numerically fit to the data using $(1/\pi)\log(\pi C/2) = 1.18$.

where for convenience we have redefined the linear parameters to be precisely the squares of the real and imaginary eigenvalues

$$\lambda^2 = c, \quad \omega^2 = (\eta - c).$$

Upon setting $x = (u, u', v/\sqrt{\omega}, v'\sqrt{\omega})$ we obtain that (4.2) is in the form of the above analysis with

$$H = \frac{1}{2}(x_2^2 - \lambda^2 x_1^2) + \frac{\omega}{2}(x_3^2 + x_4^2) + \frac{r}{2\omega}x_1 x_3^2 + \frac{s}{2\sqrt{\omega}}x_1^2 x_3 + \frac{q_1}{3}x_1^3 + \frac{q_2}{3\omega\sqrt{\omega}}x_2^3. \quad (4.3)$$

This ODE Hamiltonian can be shown to have the same non-quadratic parts as the Hamiltonian density function for the PDE system (4.1). Note that (4.3) has no $\mathbb{Z}_2$ symmetry hence we take the result from Section 3.1 above.

From (4.3) we note immediately that the Birkhoff signature is positive and that hence formula (3.23) applies (with $-\omega$ replaced by ω). Figure 4 shows the result of comparing this formula with the computation of BSs of (4.2) using a numerical shooting method for the particular case $r = s = q_1 = q_2 = 1$ We do not attempt to make an explicit calculation of C, but simply use it to fit the formula to the numerical data. The agreement between the asymptotics and numerics is nevertheless spectacular.

4.2 Third-harmonic generation

As an example of a system with odd symmetry we consider the model describing so-called type-I third harmonic generation in an optical medium, given the presence of both self- and cross-modulation. A system of PDEs modelling such a process in 1+1 dimensions can be written as follows, see [38,45,36],

$$
\begin{aligned}
& i\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial x^2}-u+\left(\frac{1}{9}|u|^2+2|w|^2\right)u+\frac{1}{3}u^{*2}w=0, \\
& i\sigma\frac{\partial w}{\partial z}+s\frac{\partial^2 w}{\partial x^2}-\alpha w+(9|w|^2+2|u|^2)w+\frac{1}{9}u^3=0.
\end{aligned}
\tag{4.4}
$$

Here u and w are the fundamental and third harmonics, respectively, x is the transverse co-ordinate and z is the propagation distance. The parameter α measures the nonlinearity induced shift in the propagation constant and is also dependent on the quality of wave-vector matching between the harmonics. The positive constant σ is the ratio of second-order group velocity dispersions for the first and the third harmonics, and $s=\pm 1$. The Hamiltonian of the PDE system (4.4) is given by

$$
\begin{aligned}
\mathcal{H}= & \int_{-\infty}^{+\infty}\left\{\left|\frac{\partial u}{\partial x}\right|^2+s\left|\frac{\partial w}{\partial x}\right|^2-\frac{1}{18}|u|^4-\frac{9}{2}|w|^4-\right. \\
& \left.-2|u|^2|w|^2-\frac{1}{9}\left(wu^{*3}+w^*u^3\right)+|u|^2+\alpha|w|^2\right\}dx
\end{aligned}
\tag{4.5}
$$

and is fully odd-symmetric, i.e. invariant under $(u,w)\to-(u,w)$.

Stationary solutions to Eqs. (4.4) are real and are described by the following system,

$$
\begin{aligned}
& u''-u+\left(\frac{1}{9}u^2+2w^2\right)u+\frac{1}{3}u^2w=0, \\
& s\,w''-\alpha w+(9w^2+2u^2)w+\frac{1}{9}u^3=0.
\end{aligned}
\tag{4.6}
$$

The Hamiltonian of the ODEs (4.6) is given by

$$
H=\frac{1}{2}\left[(u'')^2-u^2\right]+\frac{1}{2}\left[s\,(w'')^2-\alpha w^2\right]+\frac{1}{4}\left(\frac{1}{9}u^4+9w^4\right)+\frac{1}{9}u^3w+u^2w^2.
\tag{4.7}
$$

The relation between the non-quadratic parts of the Hamiltonians (4.5) and (4.7) has the assumed form (2.5) with $k=-2$. The origin of (4.6) is a saddle-centre equilibrium if $s\alpha<0$ and the singular limit is $|\alpha|\to\infty$. Upon setting $x=(u,u',|\alpha|^{1/4}w,|\alpha|^{-1/4}w')$, the Hamiltonian (4.7) takes the form:

$$
H=\frac{1}{2}(x_1^2-x_2^2)-\frac{\omega}{2}(x_4^2+x_3^2)+\frac{1}{36}x_1^4+\frac{1}{9|\alpha|^{1/4}}x_1^3x_3+O\left(\frac{1}{\alpha}\right).
\tag{4.8}
$$

where $\omega=-s\sqrt{|\alpha|}$ and $\lambda=1$. Note that the last term in H can be considered as a weak perturbation to the integrable limit when $|\alpha|\gg 1$. Comparison between Hamiltonians (2.10) and (3.28) yields $a=1/9$ and $\alpha_1=1/(9|\alpha|^{1/4})$, thus $C=1$ in (3.32).

From the result (3.42) it follows that for the case of anomalous dispersion

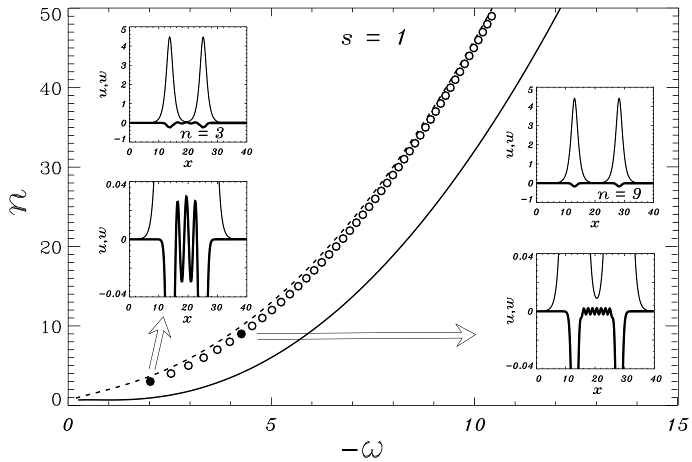

Fig. 5. Even bound states in the THG system with $s = 1$. The solid line corresponds to formula (3.43) with $C = 1$, and dots represent numerical results. The dashed line fits the numerical data after taking $\widetilde{C} = 0.03$. The insets show examples of BSs, in which thin and thick lines correspond to the fundamental wave u and its harmonic w respectively.

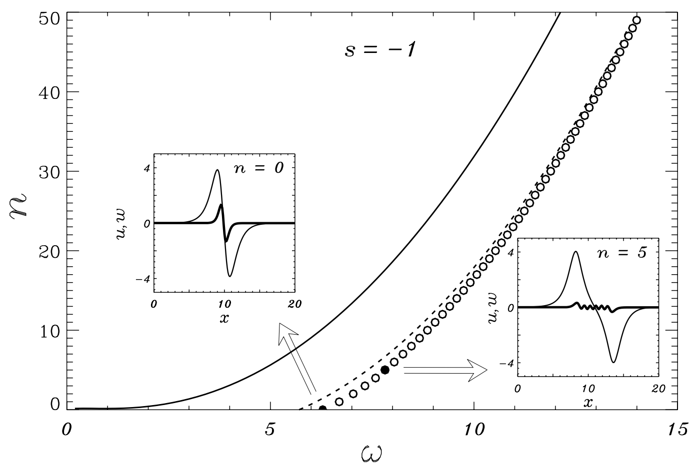

Fig. 6. Antisymmetric bound states in THG system. Data presented as in Figure 5, where now the dashed line represents the fit $\widetilde{C} = 190$.

($s = 1$) one shall expect BSs with an even fundamental wave (u-component), whereas for normal dispersion only odd fundamental components can lead to formation of a BS. Figure 5 represents results for the case of negative mismatch, $s = 1$, for which the Birkhoff signature is positive, i.e. $\omega < 0$. Note that our theory successfully predicts the symmetry properties of the BSs. Discrepancy between the theoretical prediction (3.43) and the numerical results is due to the fact that we have not done beyond all-order asymptotics to estimate the true constant C appearing in the expression for the tail radiation (3.31). The discrepancy can be eliminated by taking $\widetilde{C} = KC$, $K = 0.03$. Figure 6 represents result related to the case of anomalous dispersion and positive mismatch, $s = -1$, $\omega > 0$.

As an example of a dynamical system which is symmetric under S_3 defined in (3.44), we consider a model for type-I second harmonic generation in optical media. See, for example, [39,5] for the derivations of approximate PDEs that model such phenomena. We shall take the following simplified form in 1+1 dimension:

$$\begin{aligned} & i\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - v + v^* w = 0, \\ & i\sigma\frac{\partial w}{\partial z} + s\frac{\partial^2 w}{\partial x^2} - \alpha w + \frac{1}{2}v^2 = 0, \end{aligned} \tag{4.9}$$

where v and w are the fundamental and second harmonics respectively, x is the transverse co-ordinate, z is the propagation distance.The positive constant σ is the ratio of second-order group velocity dispersions for the first and the third harmonics, the parameter α has the same meaning as in the previous section, and again $s = \pm 1$.

Stationary solutions to Eqs. (4.9) are real and described by

$$\begin{aligned} & v'' - v + wv = 0, \\ & s\, w'' - \alpha w + \frac{1}{2}v^2, \end{aligned} \tag{4.10}$$

the origin of which is a saddle-centre equilibrium if $s\alpha < 0$. Relation (2.5) between the non-quadratic parts of the Hamiltonians of PDEs (4.9) and ODEs (4.10) is valid for $k = -2$. Upon setting $x = (u, u', |\alpha|^{1/4}w, |\alpha|^{-1/4}w')$, the Hamiltonian of ODEs (4.10) is approximated in the limit $\alpha \to \infty$ by the expression

$$H = \frac{1}{2}(x_2^2 - x_1^2) - \frac{\omega}{2}(x_4^2 + x_3^2) + \frac{1}{2|\alpha|^{1/4}}x_1^2 x_3 + \frac{1}{8\alpha}x_1^4 + \text{h.o.t}, \tag{4.11}$$

where $\omega = -s\sqrt{|\alpha|}$. and is symmetrical with respect to S_3. Note that the integrable limit for this model is large $|\alpha|$, for $w \simeq v^2/(2\alpha)$, which effects are are described by the last term in (4.11). Comparison between Hamiltonians (2.10) and (3.45) yields $a = 1/(2\alpha)$, $\alpha_1 = 1/(2|\alpha|^{1/4})$, and thus the estimate (3.49) gives $C = 1$. Taking into account the requirement that $a > 0$ for solitons to exist and that $s\alpha < 0$ in order to be in the saddle-centre parameter region, one concludes that only $s = -1$ can lead to formation of BSs. From the result (3.53) it follows that for normal dispersion one should expect BSs with an odd fundamental component. These predictions are in agreement with the results of numerical shooting, see Fig. 7.

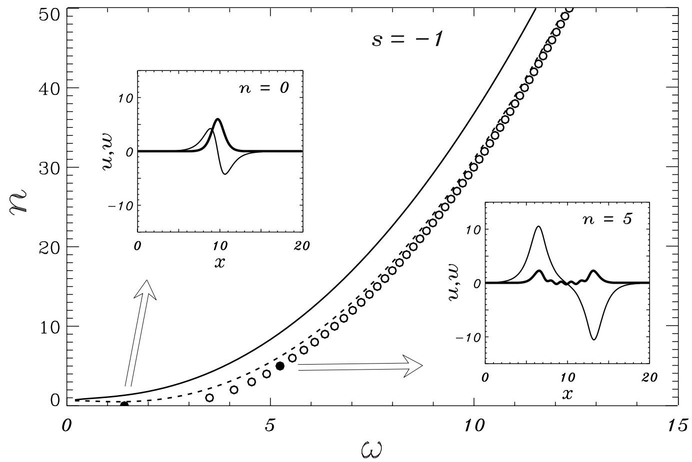

Fig. 7. Bound states due to second harmonic generation. The solid line corresponds to formula (3.54) with $C = 1$, and dots represent numerical results. The dashed line fits the numerical data after taking $\widetilde{C} = 3.5$. The insets show examples of BSs, in which thin and thick lines correspond to the fundamental wave v and its harmonic w. respectively.

4.4 *An extended massive Thirring model*

The next example is the model of an extended massive Thirring model, including second-derivative terms, proportional D below, introduced in [16]. See [14] and references therein for the physical significance of these additional terms to the generalisation of the massive Thirring model, proposed in [19,1] in order to describe solitons in an optical media equipped with a Bragg-grating. The PDEs take the form

$$\begin{aligned} iu_t + iu_x + Du_{xx} + (\sigma|u|^2 + |v|^2)u + v &= 0, \\ iv_t - iv_x + Dv_{xx} + (\sigma|v|^2 + |u|^2)v + u &= 0, \end{aligned} \tag{4.12}$$

where $D > 0$. After reduction to steady states via $u(x,t) = e^{i\chi t}U(x)$, $v(x,t) = e^{-i\chi t}U(x)$, further scalings enable one to assume without loss of generality that $\sigma = 0$ and $U = V^*$. Hence we obtain the single complex ODE

$$DU'' + iU' + \chi U + U|U|^2 + U^* = 0, \tag{4.13}$$

which has Hamiltonian

$$H = D|U'|^2 + \chi|U|^2 + |U|^4 + \frac{1}{2}(U^2 + (U^*)^2), \tag{4.14}$$

is reversible under

$$R_t : (U, U') \to (U^*, -(U')^*)$$

and has odd symmetry $S : (U, U') \to -(U, U')$.

Now, the linearisation of (4.13) shows that eigenvalues Λ satisfy

$$D^2\Lambda^4 + (2\,D\chi + 1)\,\Lambda^2 + \chi^2 - 1,$$

from which we note that the origin is a saddle-centre if $|\chi| < 1$. The singular saddle centre limit is $\chi \to 1$. In what follows we shall take $D = \text{const.} = 1.44$. Then it is straightforward to compute , in the notation of the previous section, that

$$|\Omega| = \omega/\lambda = \frac{1.905259}{(\chi-1)^{1/2}} + O(\chi-1)^{3/2}.$$

Furthermore, we note the Birkhoff signature $-\omega$ is negative. To see this, decompose into real and imaginary parts via $U = u_r + iu_i$. Then the quadratic part of the Hamiltonian (4.14) may be written as

$$H^{lin} = (\chi - 1)\left[u_i^2 - \frac{\chi+1}{\chi-1}u_r^2\right] + D\left[(u_i')^2 + (u_r')^2\right] \tag{4.15}$$

where, in order to respect the requirement that the reversibility R_t acts to reverse momenta, the canonical variables q and p must be chosen via

$$q := (q_1, q_2) = (u_r, u_i')T_1, \qquad p := (p_1, p_2) = (u_i, u_r')T_2$$

for some 2×2 transformation matrices $T_{1,2}$. Now, since we have that $-1 < \chi < 1$ and $D > 0$, it is immediate to see from (4.15) that the Birkhoff signature is negative. Hence, according to formula (3.43) there will be a family of bound states that are SR-symmetric. That is the $\mathrm{Re}(U)$-component will be odd and the $\mathrm{Im}(U)$-component even.

Figure 8(a) shows the comparison between the formula (3.43) and the numerical computation of boundstates for this example, using an approximate numerical fit for the constant C. Again we see excellent agreement. Figure 8(b) plots the squared amplitude of the boundstates as $\chi \to 1$. Since the rate of convergence to the singular limit is so slow we have computed only up to $n = 24$ for which $\chi = 0.94139$. For χ-values beyond that we have plotted two times the squared amplitude of a computation of a QS (whose tail radiation at these parameter values was approximately the sign of numerical precision). We see that the amplitude of the BSs tend quadratically to zero as $\chi \to 1$.

We end this section by illustrating how in a two-parameter problem, such as this one, the curves of BSs that accumulate algebraically on the singular limit, may form connected branches with the BSs that accumulate exponentially on a curve of fundamental ESs. For, in this model, it was found numerically in [16] that there are three curves in the (χ, D)-plane at which fundamental (1-pulse) ESs exist. Figure 9 shows the results of a two-parameter continuation of two-humped BSs. Note that the exponential convergence in the theory of [42] is proportional to $\exp(\lambda/\omega)$ which becomes slower in this example as $\chi \to 1$, which is the singular limit. This can be seen in the figure where the parameter

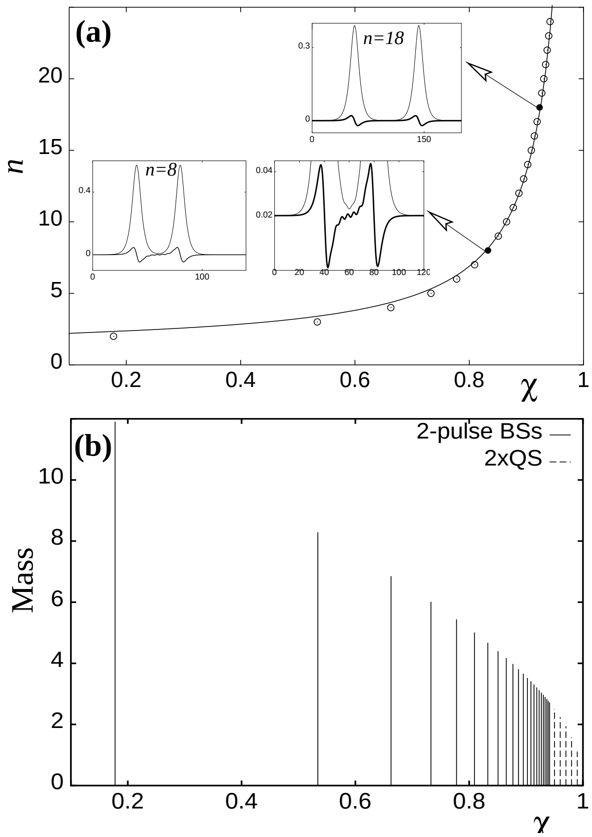

Fig. 8. Bound states in the massive Thirring model. Panel (a) shows, in analogy with the previous examples, a comparison between the numerical (circles) computation of BSs and the appropriate asymptotic formula, (3.43), for which we have used the numerical constant $(1/\pi)\log(\pi C/2) = 0.3$. In the inserts, the thick line is $\mathrm{Re}\, U$ and thin line is $\mathrm{Im}\, U$. (b) Shows the Mass of the BSs defined by $\int |U|^2 dx$.

values at which the two-humped boundstates (dashed lines) accumulate on those for the fundamental ESs (solid lines) at a rate which becomes slower as the top of the figure is approached. At the same time, the family of dashed lines converges vertically to the limit $\chi \to 1$ at a much slower rate.

4.5 The fifth-order KdV equation

Our final example is the generalised 5th-order Korteweg–de Vries (5thKdV) equation (also known as the Kawahara equation [32]) which is a PDE model for waves on water of finite depth under the influence of gravity and surface tension, see e.g. [51,29,33,47]. The most general model may be written as

$$u_\tau + \alpha u_{xxxxx} - \beta u_{xxx} + \gamma u u_x + \mu u u_{xxx} + 2\mu u_x u_{xx} + \delta u^2 u_x = 0, \qquad (4.16)$$

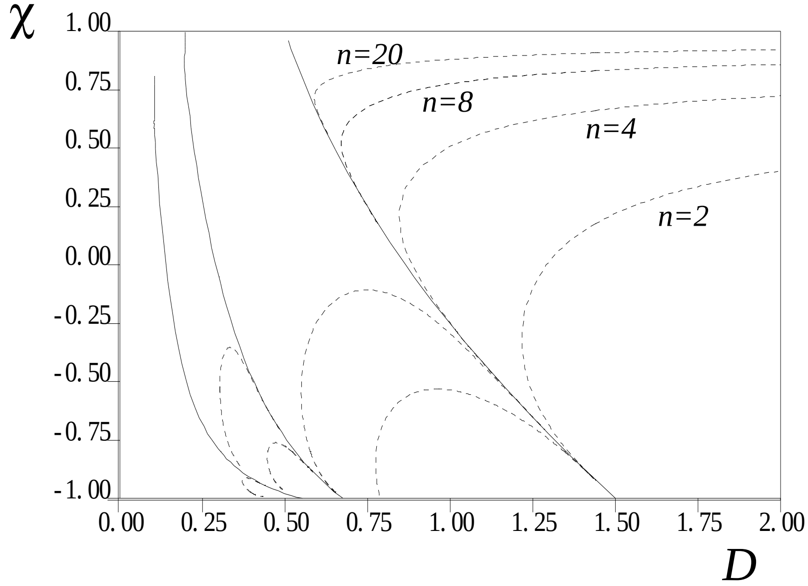

Fig. 9. Continuation in the (D, χ)-plane of a few of the R_t-symmetric BSs found for the model (4.13) (dashed lines) and the three fundamental ESs (solid lines) found by [16]

where different nonlinear coefficients γ, μ and δ arise in different applications. The PDE has Hamiltonian structure

$$u_\tau = \frac{d}{dx}\nabla\mathcal{H}(u), \quad \mathcal{H} = \int_{-\infty}^{\infty}\left(\frac{\alpha}{2}u_{xx}^2 + \frac{\beta}{2}u_x^2 + \frac{\gamma}{6}u^3 - \frac{\mu}{2}uu_x^2 + \frac{\delta}{12}u^4\right)dx. \tag{4.17}$$

One finds solitary-wave solutions of (4.16) by setting $u(x,\tau) = y(t = x - c\tau)$ and noting that the equation then becomes a total derivative with respect to t. Integrating once and choosing the constant of integration to be zero we obtain an ODE of the form

$$\alpha y'''' - \beta y'' + cy + \frac{\gamma}{2}y^2 + \frac{\mu}{2}\left[(u')^2 + 2uu''\right] + \frac{\delta}{4}y^3 = 0, \tag{4.18}$$

where a prime denotes differentiation with respect to t. Viewed as a four-dimensional dynamical system, (4.18) is reversible under

$$R : (y, y', y'', y''') \to (y, -y', y'', -y''')\ . \tag{4.19}$$

Taking the linearisation of (4.18), we note that the origin is a saddle-centre when $c < 0$ where the eigenvalues are given by $\pm\lambda$ and $\pm i\omega$, where

$$\omega^2 = \frac{\sqrt{\beta^2 - 4\alpha c} - \beta}{2\alpha}, \qquad \lambda^2 = \frac{\sqrt{\beta^2 - 4\alpha c} + \beta}{2\alpha}.$$

Hence the singular saddle-centre limit we are interested in is $a \to 0^-$ for $\beta > 0$, assuming without loss of generality that $\alpha > 0$.

Several different cases are known in which (4.18) supports fundamental ES solutions [47]. For example, in [11] the case $\alpha = 2/15$, $\mu = 1$ and $\delta = 0$ and $\gamma = 3$ is considered and an exact sech^2-solution is found along the curve $c = \frac{2}{5}(2b+1)(b-2)$, for $c < 0$. This was found to be the first among a discrete family of fundamental ESs, but crucially no multi-pulse ESs were found. Malomed & Grimshaw [26] attempted to find BS of ES in the limit $c \to 0$ for the simplest 5th-order KdV equation (with $\mu = \delta = 0$). For the same equation, Calvo & Akylas [9] found boundstates of QSs (with *non-zero* tail radiation) in this limit by a similar asymptotic method. Their calculations also imply that truly localised that BSs cannot exist. This is also reflected in the rigorous result due to Amick & McLeod [4] (see also [28]) that no symmetric homoclinic solutions to zero exist for $a < 0$ (either fundamental or multi-pulse) in an ODE which is equivalent to (4.18) with $\mu = \delta = 0$.

System (4.18) is Hamiltonian, with Hamiltonian function

$$H = \frac{\alpha}{2}\left[(y'')^2 - 2y'y'''\right] - \frac{c}{2}y^2 + \frac{\beta}{2}(y')^2 - \frac{\gamma}{6}y^3 - \frac{\mu}{2}\left[y(y')^2 + y^2y''\right] - \frac{\delta}{12}y^4. \quad (4.20)$$

Note that each of the non-quadratic terms in this Hamiltonian is the same as in the PDE Hamiltonian density function except the term proportional to μ. This discrepancy is related to the fact that kinetic terms in the Hamiltonian can not be separated from the potential part. As a consequence the model (4.16) is beyond the framework of our approach as we do not satisfy the criterion (2.5). Nevertheless, the theory developed throughout the paper successfully explains the negative result for existence of multi-pulse ESs in the model (4.16).

We now show that in fact, according to our asymptotic theory no BSs of ESs can exist in the limit $c \to 0$, for any set of nonlinear parameters γ, δ or μ, because 5th-order KdV model (4.18) has negative Birkhoff signature. For simplicity just consider the linear part of the equation, and without loss of generality let $\alpha = 1$. A particular choice of canonical co-ordinates that puts the linear part of (4.18) into Hamiltonian form is

$$q_1 = y, \quad q_2 = -y'', \quad p_1 = -y''' + \beta y', \quad p_2 = y',$$

where we note that the reversibility (4.19) changes the sign of the momentum variables $p_{1,2}$ as it should. In these co-ordinates, the linear part of H reads

$$H^{lin} = -\frac{c}{2}q_1^2 + \frac{1}{2}q_2^2 - p_1p_2 + \frac{\beta}{2}p_2^2 \quad (4.21)$$

which we can write as a sum of quadratic forms in the co-ordinates and momenta separately

$$H_{lin} := qVq^{\mathrm{T}} + pMp^{\mathrm{T}}, \quad \text{where} \quad p = (p_1, p_2), \quad q = (q_1, q_2).$$

Now, note from the normal form (2.6) an alternative characterisation of the Birkhoff signature. The signature is positive when the matrix M is definite while V is indefinite, whereas the signature is negative if the opposite conditions hold. Hence, looking at the form of (4.21) we see that the Birkhoff signature is negative under this definition in the parameter region of interest (namely $c < 0$, $\beta > 0$) since V is negative definite, whereas M is indefinite. Hence, since this system obeys no form of $\mathbb{Z}_2$ symmetry, then according to theory of Section 3.1 above, there should be no multi-pulse embedded solitons, which is indeed borne out by the previous literature.

5 Conclusion

In this paper we have reached a simple conclusion regarding the existence of two-humped embedded solitons in coupled or higher-order Hamiltonian nonlinear-wave equations in 1+1 dimensions. It is known that convergence in the parameter space of two-humped solitons onto fundamental embedded solitons is exponential. In contrast, our results show that the similar convergence of two-humped solitons onto the singular limit is algebraic. Moreover, whether such 'bound state' two-humped solutions exist at all, and whether they are odd or even is governed by the simple sign condition. It includes the Birkhoff signature of the quadratic part of the Hamiltonian together with the symmetry supported by the entire Hamiltonian. It is known that the rate of exponential convergence onto a regular embedded solitons is governed solely by the eigenvalues of the saddle-centre. Unlike this, the rate of convergence in the saddle-centre singular limit is defined by more ingredients. But by using a general approach, based on Hamiltonian normal forms, we have shown that the leading two terms of the formula governing this convergence (Equations, (3.23), (3.43) or (3.54) depending on the symmetry type) is *independent* of the model. It is only the third-order terms that require the model-specific calculation of the beyond-all-order coefficient of the minimum tail radiation of quasi-solitons.

Our results concern only two-humped solutions, but the methodology is easily extendible to deal with arbitrary N-pulses, as in [42,9]. Also, we have confined ourselves to the case where the steady-state ODEs are in $\mathbb{R}^4$. In principle though, the results are extendible to arbitrary even-dimensional Hamiltonian reversible systems, via centre manifold theory, in the case where $\pm\lambda$ and $\pm i\omega$ are the eigenvalues of the origin with real parts closest to the imaginary axis.

Even, formally, the results may extend to infinite dimensions, such as optical solitons in more spatial dimensions, or the classical Euler-equation formulation of waves on water of finite depth in the presence of surface tension. For the latter, it is known that the steady-state problem can be characterised as an infinite-dimensional Hamiltonian system, with much of the same structure as the examples considered here, and which is amenable to centre-manifold reduction (see e.g. [3,30,41,50,37,27]).

Acknowledgments

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